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THE CONTENT OF A SECOND COURSE IN CALCULUS.[1]

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The second course in calculus, as presented in different colleges and by different instructors, varies considerably in content. In view of its position on the one hand as often the culmination of the study of mathematical analysis both for students specializing in science and engineering and for students who will be teachers of mathematics in our secondary schools, and on the other hand as a necessary prerequisite for much of the later work in mathematics, it merits careful consideration. A general discussion of the course at a meeting like this should prove stimulating to all of us who have been faced with its difficulties; and it is my desire to learn the views of other teachers as much as to argue my own views that has led me to speak on this subject to-day.

Before considering the content of the course we should think of what its general aim should be. This general aim in turn depends upon the students who are expected to take the course, their preparation and their requirements. As to preparation, the minimum may be assumed to consist of courses in trigonometry, college algebra, analytical geometry and a three-hour year course in calculus. This is doubtless quite generally exceeded, but closely approximates the most common actual prerequisites for the second calculus course; and it is not desirable to assume a much better preparation. In my own classes, for instance, many of the students have had, or are taking, a course in mechanics, but I find it impossible to assume that course as a part of the student's preparation. At best the student's outlook on the field of mathematics is very restricted, and even his working knowledge of calculus is limited to the very simplest portions.

[1] This paper is a revision of one presented to the Mathematical Association of America at the meeting in Chicago on December 27, 1917.

As to the student's requirements, these are best discussed by dividing the students into three groups: first, those who are specializing in science or engineering; second, those for whom this is the last course in analysis and who will presumably have no direct use for the calculus in their subsequent work; and third, those for whom this is a preparation for more advanced mathematics.

For the first group the ability to use mathematics easily and vigorously is clearly the great need. Mathematics for them is primarily a tool, and secondarily a mental discipline; and the general utility of any topic might well be the basis for deciding whether it should be given. For them it is less important to give painstaking proofs of important theorems than to emphasize the importance of those theorems by using them in the solutions of problems which they recognize as practical problems or applications of mathematics. And we may furthermore suggest that the mental discipline is as great in a careful use of a theorem as in the proof of it.

In the second group are prospective teachers and students who may be described as taking mathematics for its cultural value. They are, in general, students who are making mathematics their major subject in college, and will accordingly take other courses of about the same grade, probably in projective geometry or the theory of equations. In these latter subjects mathematics as a systematic, logical development of certain lines of thought is well illustrated, but mathematics as a tool for the discussion of the physical world is largely, if not completely, neglected. It is important that these courses be given in this way; but it is also important that the student shall appreciate the great utility of mathematics. The topics which one associates with the calculus are, at least in part, well adapted to be given so as to show this utility, and accordingly it seems that this aspect of the subject should receive a considerable emphasis. Moreover, the students of the second group require less formal work than those of the first group; they need instead to cover a wide range of topics with only sufficient formal drill to be able to follow the course readily.

Concerning the third group, composed of future mathematicians, there will doubtless be a diversity of views as to their needs. My own feeling is that it is too early in the student's career to introduce the critical attitude of higher mathematics, that it is far better at this stage of the student's development to develop his technique in handling the great algorisms of calculus, and at the same time, as far as possible, to give him a broad outlook on the great classical problems of analysis. The student needs a background before entering on any detailed study of the fundamental properties of continuous functions, for example, and it is especially within the province of this second course in calculus to provide that background. A study of those fundamental properties should follow this course instead of being a part of it. Moreover, it seems to me important that the student should at this time get some notion of a considerable number of topics rather than go more extensively into some one subject, like differential equations, for instance. The student will presumably go more deeply into each of the topics later, and the advantage of having some preliminary notion of the subject in each case and of

a number of related subjects should much more than make up for the time apparently lost in repetitions.

It seems, then, that we have three things that we should aim to do: first, we should aim to develop the student's technique in using with some freedom as many as possible of the great algorisms we naturally associate with the calculus; second, we should aim to present as far as possible problems which the student will recognize as applications of mathematics; and third, we should aim to give as broad an outlook as possible on the classical problems of analysis.

Granting that these should be our aims, there arise a number of questions as to methods of procedure and questions as to what should be the actual topics taken up. First of all, it is obviously undesirable to try to build up our calculus from a few simple postulates in a manner satisfactory from the point of view of formal logic, for time would not permit, even if we assumed that the student could appreciate such a method. On the contrary, it is desirable to make a great number of assumptions, subject of course to the condition that they be consistent, and further that the student will agree to their validity intuitionally.[1] The more we reflect on the fundamental postulates and elementary theorems involved in questions in limits and continuity, the more important it appears that the specialist in mathematics should eventually go into these questions deeply, but also the more important it seems that in the earlier stages of his training he shall make a free use of his geometric intuitions in such questions under the guidance of a more sophisticated intuition. I would therefore urge that we use our geometric intuitions with some freedom in this course, the instructor holding himself responsible for their correctness.

Furthermore is it wise to insist on giving proofs of all, or even of approximately all, of the theorems we may wish to state, even when those theorems do not appear intuitional to the student? The fact that mathematics is built up as a matter of logical proof from a set of axioms gives the subject a special claim for our attention, but is it advisable to require our students to go through the proofs of all, or of approximately all, of the theorems they may ever want to use?

My belief is that we would do better if we took greater freedom in stating theorems without proof, and spent more time in giving our students an idea of the topic under discussion in its entirety, incidentally requiring them to build higher through a more frequent acceptance of the dicta of others. This involves no necessary loss in the training of the student's reasoning powers, for the proofs that are given need be given no less carefully. The student may feel the lack of the completeness which is one of the charms of mathematics, and this would be serious if it came too early in his course, but it seems to me that the gains far exceed the losses if this policy is adopted in the course we are discussing.

As to the standard of rigor to be maintained in the proofs which are given, it is clear, with our expressed aims, that there should not be any radical change from the standards with which the student is already familiar. It is more

[1] See E. H. Moore: "On the Foundations of Mathematics," *Bulletin of the American Mathematical Society*, Vol. 9, 1902–1903, for remarks on the use of our intuition in elementary mathematics.

important to give arguments in a form which will be readily understood by the student and which are essentially complete and exact, even if not perfectly satisfactory from the point of view of formal logic, than to lay more stress on the logical minutiæ of the proofs. Geometrical intuitional arguments which are essentially conclusive should be used with freedom whenever they will aid in clearness of presentation. But arguments which involve actual misstatements, or which the best students may easily see to be illogical or fundamentally incomplete, should of course be avoided. Furthermore, carelessness in such things as dividing by zero, neglecting one of the square roots of a quantity, or assuming the converse of a proposition should not be permitted. The last error particularly is to be avoided, for this is more than a mathematical error—it is an element of daily reasoning which it is perhaps as important for the student to appreciate as any of the mathematical theory which he will learn.

Another thing which should perhaps be mentioned is the importance to be attached to problems in the course, and the character of the problems. Professor Osgood has said:[1] "The process by which the youth actually acquires the ideas of the calculus is to a large extent and essentially through formal work of substantial character. In order to attain this end however the formal work must appear to him as having for its direct object the power to solve some of the real problems of pure and applied mathematics, and those problems must be kept before his eye." My own experience as student and teacher bears out these statements. In the presentation of the course the solution of problems should constitute a vital part, a superficially more important part than the development of the theory. In this way the student's technique is developed, he obtains an appreciation of the utility of the subject, and he may be led to a consideration of many of the great old problems.

It is my purpose to take up in a moment an explicit list of topics suitable for the course under consideration, chosen in an attempt to meet the aims we have stated. Owing to differences in the first course as given at different times and places it is found necessary to include some topics which are often adequately treated in the first course in the calculus. One would naturally wish to avoid repetition except in the case of the most fundamental notions, where a review is worth while, but these topics are included so that they will not be overlooked in both courses in calculus. Recalling that we assume as a prerequisite only a three-hour year course in calculus, we see why there will be a considerable overlapping of topics with those often considered in a first course or in other courses of about the same grade.

The topics will be taken up in the order in which they might well be given to the class, dealing with partially known and simpler topics first, and leading to new and rather more difficult topics last,—following this out as far as seems compatible with a natural desire to associate closely related topics.

As an introductory chapter we need a brief review of the ideas of function,

[1] In his presidential address to the American Mathematical Society, "The Calculus in Our Colleges and Technical Schools," April 27, 1907; see the *Bulletin of the American Mathematical Society*, Vol. 13, 1906–1907, pp. 449–467.

limit, continuity, derivative, and integral. It is more important here to call attention again to the fundamental importance of the idea of function in applications of mathematics than to attempt any refinements in the definitions of limits and continuity. It is desirable to include a classification of the elementary functions, a discussion of their discontinuities, and a working test for locating discontinuities of a function of a function.[1] By introducing hyperbolic functions here we can both make it possible to use these important functions later and at the same time provide some new material for practice in the old methods of differentiation and integration.

A chapter on the simplest types of differential equations of the first and second orders can be given as a simple application of differentiation and integration. A discussion of families of curves, orthogonal trajectories, and the solution of numerous problems in geometry and mechanics will show the utility of the subject to the student. The importance and significance of the constants of integration should receive emphasis as well as the formal solution of equations.

This may be followed by a chapter leading up to and including Taylor's Theorem with Remainder. Starting from Rolle's Theorem and the simpler Law of the Mean as geometrically evident, we may prove the extension of the latter theorem to the quotient of two functions, and establish l'Hospital's Rule for the evaluation of indeterminate forms.[2] These theorems may be applied to the geometrical problem of contact of plane curves and to the problem of finding polynomial approximations to functions. This leads directly to Taylor's Theorem with Remainder, which is illustrated with special functions and is applied in computations and applications to extremes and points of inflection.

The consideration of R_n, the Remainder Term in Taylor's Theorem, as a function of n leads to infinite series. Convergence tests, including the comparison test, Cauchy's integral test and d'Alembert's ratio test, the simplest theorems on alternating series and absolute convergence, and a statement without proof of theorems concerning power series and uniform convergence, constitute the theory of series. They may be applied to various computations and to the solution of differential equations.

The consideration of a definite integral as the limit of a sum is closely related to infinite series, and though the subject is treated in the first course in calculus, its fundamental importance makes it merit further emphasis. Taking this up now, we may include approximation formulas, evaluations by ingenious devices and by infinite series as well as by the simple use of the indefinite integral method. Comparison theorems and mean value theorems for definite integrals may be given and applied to the approximate evaluation of difficult integrals as well as to certain theoretical questions.

Improper integrals, with tests for convergence, follow the preceding. Im-

[1] The last point is one quite often overlooked and has in my experience caused many good students perplexity, finding as they do the assumption of continuity in nearly every theorem, without being able to tell whether the functions with which they deal are continuous.

[2] It is very questionable if the proof of the latter theorem for the form ∞/∞ is worth giving, considering its difficulty for the student.

portant special integrals, including the Eulerian integrals, provide applications for this topic.

A brief treatment of elliptic integrals and functions may now be introduced. Only the simplest properties and uses of these functions can be given, time not permitting a treatment of the general methods of reducing elliptic integrals to standard types, computation of the standard integrals, or a systematic development of the theory of elliptic functions.

Thus far we have considered only functions of a single variable. To introduce functions of two or more variables I find it necessary to devote some time first to solid analytics. Then partial derivatives, the total differential, change of variable, differentiation of implicit functions, Jacobians and Taylor's Series may be treated. Applications are made to small errors and to geometry of space, including tangent planes and lines, osculating planes, maxima and minima, curves on a sphere, cylinder and cone. A part of the preceding will be a review of what has been studied in the first course in calculus. Also the next topic, multiple integrals, will be partly a review. Here we may include the definition as a limit of a sum, a restatement, if necessary, of Duhamel's Theorem (or a substitute), and a variety of applications to problems in areas, volumes, mass, centroids, pressure, moments of inertia, and attraction. Line and surface integrals, Green's and Stokes's Theorem and applications form an important and interesting extension of the study of integrals, and naturally follow the preceding.

There remain a number of topics one would wish to discuss, but time will not suffice to present all of them, and it is not quite clear which should have preference. Of these, probability, calculus of variations, Fourier's Series, and a further study of differential equations could be given in the spirit of the preceding, and all have a strong claim for recognition on account of their importance in various realms of applied mathematics. Vector analysis and functions of a complex variable are less closely related to what has preceded, but are perhaps of even more fundamental importance to the physicist and engineer. On the other hand, the pure mathematician may feel a strong need for a thorough discussion of certain existence theorems or of transformation of infinite series, or of questions of double limits that arise in the calculus; but my belief is, in keeping with general aims we have stated, that these topics must be sacrificed. It is quite obvious that there will be a lack of time for an extensive treatment of any of these topics, and it would doubtless be best to attempt to discuss only two or three of them. As to which these should be, I have at present no decided opinion. Perhaps we should, on general principles, allow for that much latitude in difference of opinion, and not even try to particularize further.